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ABSTRACT

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IDENTIFICATION AND ESTIMATION IN PATH ANALYSIS WITH UNMEASURED VARIABLES

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Identification and Estimation in Path Analysis

with Unmeasured Variables

Abstract

A variety of path models involving unmeasured variables are formulated in terms of Jöreskog's (1970a) general model for the analysis of covariance structures.

Identification and Estimation in Path Analysis with Unmeasured Variables*

A variety of authors (e.g., Blalock, 1969; Costner, 1969; Heise, 1969) have applied path analysis to problems involving multiple indicators of underlying constructs. An important and often algebraically complex feature of such analysis is the determination of identifiability of model parameters. The purpose of this discussion is to demonstrate how a visual inspection of the path diagram can be used to simplify the identification question and how these problems may be formulated in Jöreskog's (1970a) general model.

I. A Single Factor Model

Consider the case of a single underlying factor (F_1) with three observed measures $(X_1, X_2,$ and $X_3)$ as shown in Figure 1.a. The factor loadings $(\rho_{X_i F_1})$ in this model equal the standardized path coefficients $(b_1^*, b_2^*,$ and $b_3^*)$, given the assumption that the residuals $e_1, e_2,$ and e_3 are independent of each other and of the factor. It is convenient, though not necessary, to assume that both measured and unmeasured variables are standardized. For heuristic purposes observed correlations will be designated with "r" and expected values of these correlations by "ρ". The expected correlations will differ from the corresponding observed correlations because of sampling and model specification errors.

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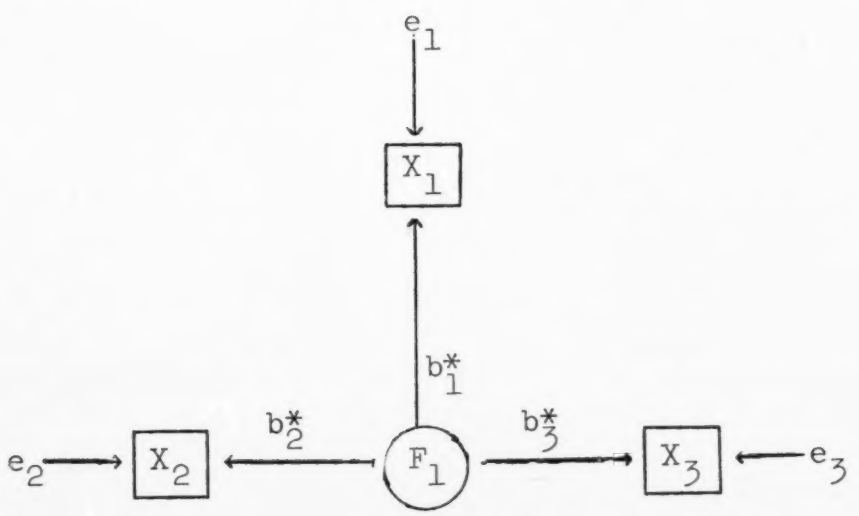


Fig. 1.a. A Single Factor Model

A path analysis of this model yields the equations:

$$\rho_{12} = b^*_1 b^*_2 \quad ,$$

$$\rho_{13} = b^*_1 b^*_3 \quad , \qquad (1)$$

and $$\rho_{23} = b^*_2 b^*_3 \quad .$$

Assuming nonzero correlations, equations (1) yield:

$$(b^*_1)^2 = \frac{\rho_{12}\rho_{13}}{\rho_{23}} = \rho^2_{X_1F_1} \quad ,$$

$$(b^*_2)^2 = \frac{\rho_{12}\rho_{23}}{\rho_{13}} = \rho^2_{X_2F_1} \quad \text{and} \qquad (2)$$

$$(b^*_3)^2 = \frac{\rho_{13}\rho_{23}}{\rho_{12}} = \rho^2_{X_3F_1} \quad .$$

Given only three observed measures the model is just identified, i.e., the observed and expected correlations are identical. With more than three measures

$$\rho^2_{X_iF_1} = (b^*_i)^2 = \frac{\rho_{ij}\rho_{ik}}{\rho_{jk}} \quad , \qquad \text{where } i \neq j \neq k \text{ and} \qquad (2a)$$

assuming $\rho_{jk} \neq 0$. If there were a causal linkage (e.g., $F_1 \rightarrow I_1 \rightarrow I_2 \rightarrow X_i$) from F_1 to X_i then $\rho_{X_i F_1}$ would be the product of the intervening path coefficients, i.e., the product of the path coefficients in the chain from F_1 to X_i would be identified. If any loading exceeded unity, the model would be rejected. When there are $m > 3$ observed measures then the loadings will be overidentified. The number of overidentifying restrictions is simply the number of distinct correlations $m(m - 1) \div 2$ less the number (m) of $\rho_{X_i F_1}$ to be estimated. Maximum likelihood or least squares estimates for over-identified models can be obtained using Jöreskog's (1970a) general method for the analysis of covariance structures. We use path analysis only to study the identifiability problem, not for estimation purposes (Hauser & Goldberger, 1970; Werts, Jöreskog, & Linn, in press).

The above analysis leads to our "rule of three": Whenever the correlations among at least three observed variables may be completely ascribed to the presence of an underlying factor, then the loadings (correlations) for each observed variable on that factor are identifiable. An important qualification is that the expected correlation between any two observed variables cannot be zero since equation (2a) would not be defined when that correlation was in the denominator. In practice, small expected correlations may lead to unstable parameter estimates, i.e., highly unreliable measures result in unreliable parameter estimates.

II. Generalizations

The Figure 1.a. model with or without intervening, unmeasured variables going from F_1 to X_i is too limited for most causal analyses. Our purpose in this section is to consider other causal patterns which satisfy the "rule of three," i.e., in which the observed correlations among three variables are

nonzero and may be ascribed to the presence of an underlying factor. Equations (1), and therefore (2), would still hold if for one of the measures (e.g., X_1) $X_1 \rightarrow F_1$ and the residual θ_1 of this regression of F_1 on X_i were independent of the other residuals e_2 and e_3, as shown in Figure 1.b.

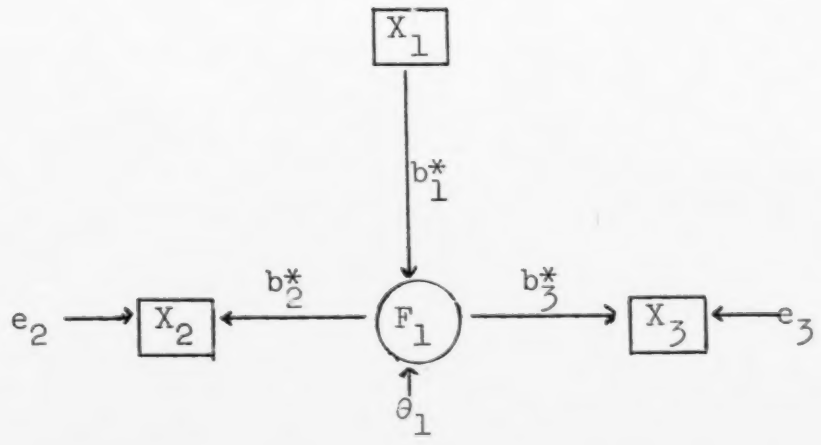


Figure 1.b.

If two observed measures influence F_1, e.g., $X_1 \rightarrow F_1$ and $X_2 \rightarrow F_1$ then it is no longer true that the correlation between these measures equals the product of the corresponding path coefficients, e.g., ρ_{12} would not in general equal $b_1^* b_2^*$.

Given that all residuals are independent, when there is an intervening variable (I_1) between X_i and F_1, the correlation between a pair of observed variables X_i and X_j will equal the product of the intervening path coefficients when $X_i \leftarrow I_1 \leftarrow F_1 \rightarrow X_j$, $X_i \leftarrow I_1 \rightarrow F_1 \rightarrow X_j$, $X_i \rightarrow I_1 \rightarrow F_1 \rightarrow X_j$, and $X_i \leftarrow I_1 \leftarrow F_1 \leftarrow X_j$; but not when two arrows point towards the same variable, e.g., $X_i \rightarrow I_1 \leftarrow F_1 \rightarrow X_j$ or $X_i \rightarrow I_1 \rightarrow F_1 \leftarrow X_j$. In general the correlation between two observed variables may be stated as the product of the intervening path coefficients whenever the causal linkage between these variables does not include a variable which is caused by two other variables, i.e., when two causal arrows point towards a variable. To identify the loadings on a factor

we need to find three observed variables which are causally linked through that factor, the linkages satisfying the above criteria.

III. Examples

A. Our first example, which corresponds to Figure 1 in Wiley and Wiley (1970), is shown in Figure 2.a.

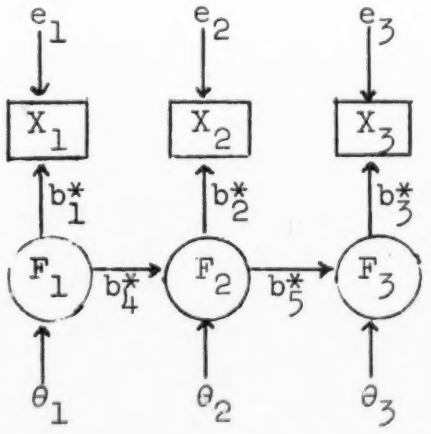


Figure 2.a.

Tracing linkages for F_2 :

$$X_1 \leftarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow X_3 ,$$

$$X_1 \leftarrow F_1 \rightarrow F_2 \rightarrow X_2 , \text{ and}$$

$$X_2 \leftarrow F_2 \rightarrow F_3 \rightarrow X_3 .$$

Since these three linkages all include F_2 and satisfy the requirements of the "rule of three" we may conclude that the factor loadings $(\rho_{X_i F_2})$, i.e., the correlations of each observed variable with F_2, are identified. Thus,

$$\begin{aligned} \rho_{X_1 F_2} &= b^*_1 b^*_4 , \\ \rho_{X_2 F_2} &= b^*_2 , \text{ and} \\ \rho_{X_3 F_2} &= b^*_3 b^*_5 . \end{aligned} \tag{3}$$

The factor loadings on F_1 are not identified because the correlation between X_2 and X_3 cannot be completely ascribed to F_1. Likewise the loadings on F_3 are not identified because the correlation between X_1 and X_2 cannot be ascribed to F_3. Jöreskog (1970b) shows that this model may be estimated by a single factor model with F_2 as the common factor and that the example may be generalized to more than three measured variables.

B. Our second example (see Figure 2.b) corresponds to Figure 4 in Costner (1969). The analysis is identical whether $F_1 \rightarrow F_2$ or $F_2 \leftarrow F_1$.

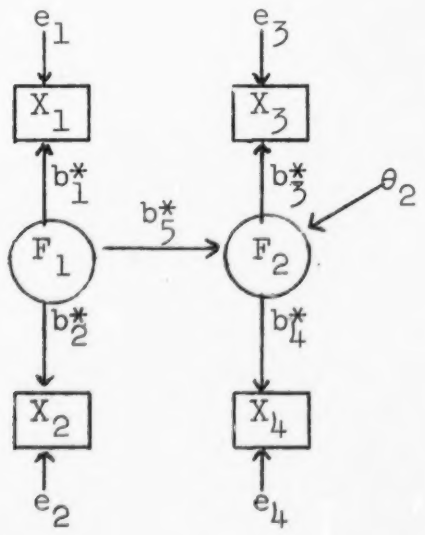


Figure 2.b.

Tracing linkages:

$$X_1 \leftarrow F_1 \rightarrow X_2 \quad , \tag{4a}$$

$$X_1 \leftarrow F_1 \rightarrow F_2 \rightarrow X_3 \quad , \tag{4b}$$

$$X_1 \leftarrow F_1 \rightarrow F_2 \rightarrow X_4 \quad , \tag{4c}$$

$$X_2 \leftarrow F_1 \rightarrow F_2 \rightarrow X_3 \quad , \tag{4d}$$

$$X_2 \leftarrow F_1 \rightarrow F_2 \rightarrow X_4 \quad , \text{ and} \tag{4e}$$

$$X_3 \leftarrow F_2 \rightarrow X_4 \quad . \tag{4f}$$

For F_1 the factor loadings may be identified by linkages 4a,b,d or by 4a,c,e, i.e., these loadings are overidentified and

$$\rho_{X_1F_1} = b_1^* \ ,$$

$$\rho_{X_2F_1} = b_2^* \ ,$$

$$\rho_{X_3F_1} = b_3^* b_5^* \ , \text{ and}$$

$$\rho_{X_4F_1} = b_4^* b_5^* \ .$$

The factor loadings for F_2 may be identified by 4b,c,f or 4d,e,f and:

$$\rho_{X_1F_2} = b_1^* b_5^* \ ,$$

$$\rho_{X_2F_2} = b_2^* b_5^* \ ,$$

$$\rho_{X_3F_2} = b_3^* \ , \text{ and}$$

$$\rho_{X_4F_2} = b_4^* \ .$$

Since b_1^* and b_2^* are identified, b_5^* is also identified by these equations.

The analysis may be complicated by assuming e_1 correlated with e_3, in which case linkage 4b would not be valid, however the conditions of the "rule of three" would still be satisfied for F_1 and F_2 and all path coefficients and correlations between errors are (just) identified. Such a model would correspond to Figure 5.a. in Costner (1969).

C. The next example, corresponding to Figure 1 in Blalock (1963), is shown in Figure 2.c. This model is basically a variation on the model of Figure 1.b.

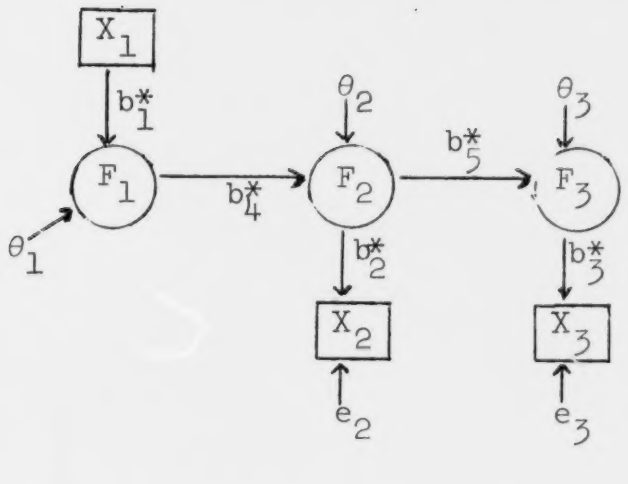


Figure 2.c.

This model differs from that in Figure 2.a. in that $X_1 \rightarrow F_1$ instead of $F_1 \rightarrow X_1$. The linkages are:

$$X_1 \rightarrow F_1 \rightarrow F_2 \rightarrow X_2 \quad ,$$

$$X_1 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow X_3 \quad , \text{ and}$$

$$X_2 \leftarrow F_2 \rightarrow F_3 \rightarrow X_3 \quad .$$

Since F_2 is in all three linkages which satisfy the "rule of three," the factor loadings for F_2 are identified and

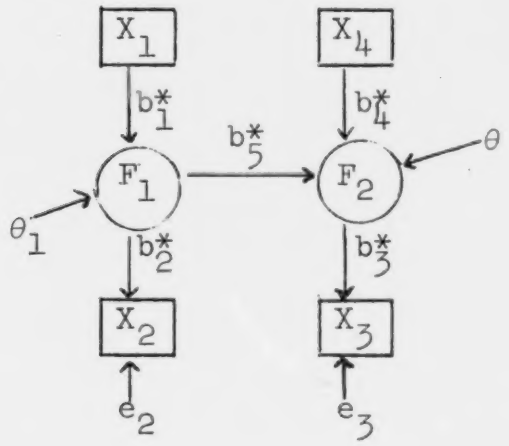
$$\rho_{X_1F_2} = b_1^* b_4^* = \sqrt{r_{12}r_{13} \div r_{23}} \quad , \tag{5a}$$

$$\rho_{X_2F_2} = b_2^* = \sqrt{r_{12}r_{23} \div r_{13}} \quad , \text{ and} \tag{5b}$$

$$\rho_{X_3F_2} = b_3^* b_5^* = \sqrt{r_{13}r_{23} \div r_{12}} \quad . \tag{5c}$$

Since r_{12} cannot be ascribed to F_3 and r_{23} cannot be ascribed to F_1 , the loadings on these factors are not identified. Our heuristic device would have been helpful to Blalock (1963) since he obtained the equations corresponding to the linkages shown above, but did not solve them for the equivalent of equations 5a,b, and c.

D. Our fourth example, shown in Figure 2.d., corresponds to Figure 2 in Blalock (1963).



θ = residual of F_2 on X_4 and F_1 .

Figure 2.d.

Tracing linkages which satisfy our rule:

$$X_1 \rightarrow F_1 \rightarrow X_2 \quad , \tag{6a}$$

$$X_1 \rightarrow F_1 \rightarrow F_2 \rightarrow X_3 \quad , \tag{6b}$$

$$X_2 \leftarrow F_1 \rightarrow F_2 \rightarrow X_3 \quad , \text{ and} \tag{6c}$$

$$X_4 \rightarrow F_2 \rightarrow X_3 \quad . \tag{6d}$$

In this model it is assumed that X_4 is independent of X_1 and X_2 . The loadings on F_1 are identified by linkages 6a,b and c and therefore:

$$\rho_{X_1F_1} = b^*_1 \quad , \tag{7a}$$

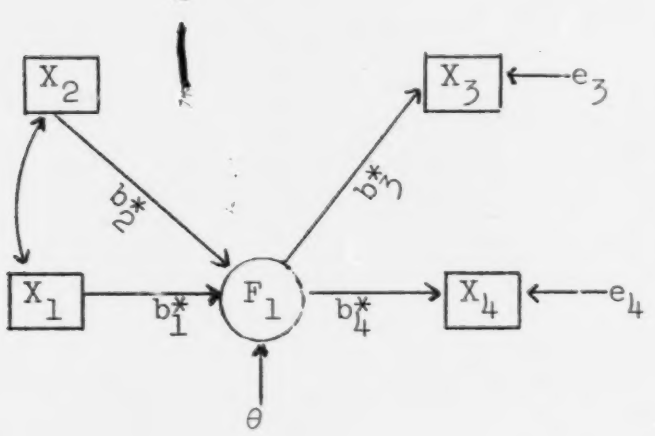
$$\rho_{X_2F_1} = b^*_2 \quad , \text{ and} \tag{7b}$$

$$\rho_{X_3F_1} = b^*_3 b^*_5 \quad . \tag{7c}$$

It is not possible to find three observed variables whose linkages satisfy our rule for F_2 , i.e., the linkage between X_1 and X_4 has two arrows

pointing at F_2 and the linkage between X_1 and X_2 does not include F_2. Since $\rho_{34} = b_3^* b_4^*$ it follows from equation (7c) that $\rho_{X_3F_1} b_4^* = \rho_{34} b_5^*$.

E. The fifth example, shown in Figure 2.e., has the special feature of two observed nonindependent variables influencing an unobserved variable. It corresponds to Figure 4 in Blalock (1969).



θ = residual of F_1 on X_1 and X_2 regression.

Figure 2.e.

When X_2 is deleted X_1, X_3, and X_4 form the model in Figure 1.b. from which we conclude that the correlations of X_1, X_3, and X_4 with F_1 are identified. Similarly when X_1 is deleted the correlations of X_2, X_3, and X_4 with F_1 are identified. Given the correlations among X_1, X_2, and F_1 the path coefficients b_1^* and b_2^* may be identified since:

$$b_1^* = \frac{\rho_{X_1F_1} - \rho_{12}\rho_{X_2F_1}}{1 - \rho_{12}^2} \quad \text{and}$$

$$b_2^* = \frac{\rho_{X_2F_1} - \rho_{12}\rho_{X_1F_1}}{1 - \rho_{12}^2} \quad .$$

F. Our last example, shown in Figure 2.f. corresponds to Figure 9.b. in Costner (1969)

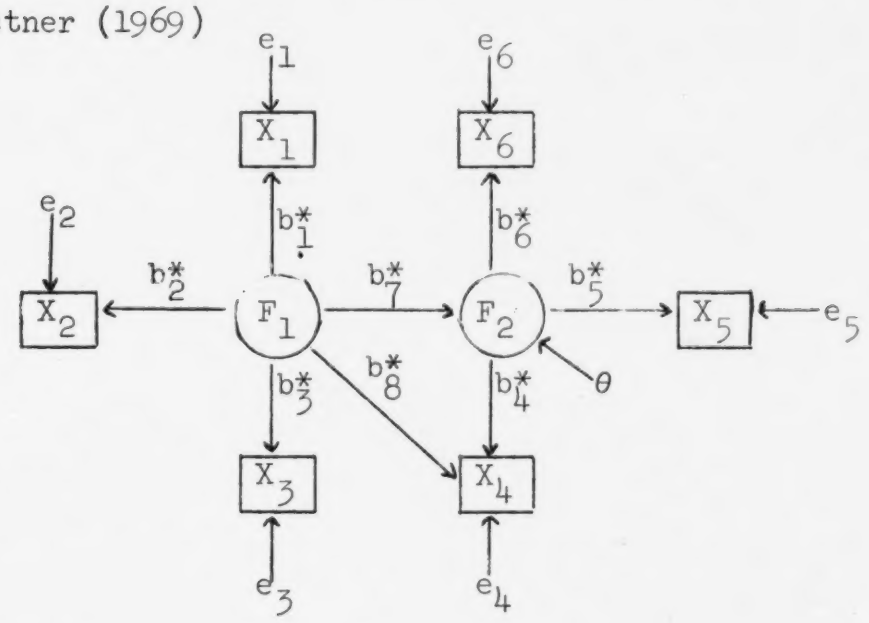


Figure 2.f.

From the analysis of the Figure 2.d. model we may deduce that when X_4 is excluded that b_1^*, b_2^*, b_3^*, b_5^*, b_6^*, and b_7^* are identified. Using the variables X_1, X_2, and X_4 we know from our analysis of the Figure 1 model that the correlation of X_4 with F_1 ($\rho_{X_4F_1}$) is identified and similarly using X_4, X_5, and X_6 we know that the correlation of X_4 with F_2 ($\rho_{X_4F_2}$) is identified. Since the correlations among F_1, F_2, and X_4 are identified it follows that the path coefficients b_4^* and b_8^*, which are functions of these correlations, are identified. As compared to Costner's (1969) rather complex algebraic analysis of this problem, it may be seen that we are satisfied in merely knowing that the model parameters are identified.

IV. Estimation

Jöreskog's (1970a) general model for the analysis of covariance structures can be used to estimate the parameters for the models discussed

above. Werts, Jöreskog and Linn (in press) discuss the use of Jöreskog's model from the perspective of path analysis. Use of the associated computer program (Jöreskog, Gruvaeus, & van Thillo, 1970) for the present purposes requires the investigator to specify a matrix Λ corresponding to the factor loadings in factor analysis; a matrix Φ which is the variance-covariance matrix of the unmeasured factors, and a matrix Θ of residual variances. The matrices B and Ψ in Jöreskog's formula are taken as the identity and zero matrix respectively.

Consider for example the model in Figure 1 in which

$$\Lambda = \begin{bmatrix} b^*_1 \\ b^*_2 \\ b^*_3 \end{bmatrix} ,$$

$$\Phi = [1] ,$$

and

$$\Theta = \begin{bmatrix} V_{e_1} & 0 & 0 \\ 0 & V_{e_2} & 0 \\ 0 & 0 & V_{e_3} \end{bmatrix} .$$

Define: $\underset{\sim}{X}$ = column vector of standardized observed variables,

$\underset{\sim}{F}$ = column vector of factors, and

$\underset{\sim}{e}$ = column vector of residuals.

In matrix terminology:

$$\underset{\sim}{X} = \Lambda \underset{\sim}{F} + \underset{\sim}{e} . \qquad (8)$$

Equation (8) is shorthand for the path equations (all variables standardized):

$$X_1 = b_1^* F_1 + e_1 \quad ,$$

$$X_2 = b_2^* F_1 + e_2 \quad , \text{ and}$$

$$X_3 = b_3^* F_1 + e_3 \quad .$$

It can be seen that Λ is the matrix of the coefficients of F_1. The parameters in the matrices specifying the model structure in Jöreskog's model are of three kinds: (1) fixed parameters that have been assigned given values; (2) constrained parameters that are unknown but equal to one or more other parameters; and (3) free parameters that are unknown and not constrained to be equal to any other parameter. In the above example the unity in Φ is a fixed parameter, whereas the b_i^* in Λ and the V_{e_i} in Θ are free parameters.

The expected variance-covariance matrix Σ for this problem is:

$$\Sigma = \Lambda\Phi\Lambda' + \Theta^2 \qquad (9)$$

where the 1 in Φ is the variance of F_1, for convenience standardized (i.e., equal to unity) and Θ^2 is a diagonal matrix whose elements are the error variances (V_{e_i}). Equation (9) should be recognized as a shorthand way of expressing all the path equations relating expected model correlations to model parameters, i.e.,

$$\Sigma = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} \quad ,$$

(where unities indicate observed variables were standardized).

Equation (9) states:

$$1 = (b_1^*)^2 + V_{e_1} \quad ,$$

$$1 = (b_2^*)^2 + V_{e_2} \quad ,$$

$$1 = (b_3^*)^2 + V_{e_3} \quad ,$$

$$\rho_{12} = b_1^* b_2^* \quad ,$$

$$\rho_{13} = b_1^* b_3^* \quad , \text{ and}$$

$$\rho_{23} = b_2^* b_3^* \quad .$$

This short description for a single model contrasts with the path analysis approach to estimation used by Costner (1969) and Blalock (1969) in the following respects:

(a) The matrix Σ of expected correlations between observed variables will differ from the actually observed matrix of correlations because of sampling and/or model specification errors. Thus we do not use observed correlations in our equations as in the usual path analysis approach. Instead, Jöreskog's program attempts to minimize the difference between observed and expected variance-covariance matrices using either a least squares or maximum likelihood approach. In large samples, assuming that observed variables are distributed normally, a chi square statistic is produced which measures the overall fit of the model to the data. Another way of gauging fit is to compare the differences between the observed and expected correlations generated by the model.

(b) The degrees of freedom (df) for the χ^2 measure are equal to the number of overidentifying restrictions. In path analysis this corresponds

to the number of different ways the path equations may be solved for each parameter. To compute the df it is necessary to count the number of distinct elements in Σ (i.e., $m(m+1) \div 2$) and subtract the number of parameters to be estimated (e.g., $b_1^*, b_2^*, b_3^*, V_{e_1}, V_{e_2}$, and V_{e_3}). There is no need to solve the path equations in Jöreskog's approach, although the identifiability must be known.

To analyze the model in Figure 1.b., we merely need to note that when X_1 and F_1 are standardized the regression of X_1 on F_1 equals that of F_1 on X_1 and the residuals are identical. Thus we may use the same estimation procedure for this model as for that in Figure 1.a. (where $\theta_1 = e_1$). Likewise the models in Figures 2.a. and 2.c. may be estimated by ignoring F_1 and F_3 and treating X_1, X_2, and X_3 as indicators of the common factor F_2.

The model in Figure 2.b. with the added feature of e_1 and e_3 correlated requires special treatment. The equations are:

$$X_1 = b_1^* F_1 + e_1 \ ,$$
$$X_2 = b_2^* F_1 + e_2 \ ,$$
$$X_3 = b_3^* F_2 + e_3 \ ,$$
$$X_4 = b_4^* F_2 + e_4 \ , \text{ and}$$
$$F_2 = b_5^* F_1 + \theta_2 \ .$$

We know that b_5^* is equal to the correlation between F_1 and F_2 so there is no need to replace F_2 by F_1 and θ_2 in the first four equations. To specify a correlation between e_1 and e_3, all residuals must be treated as factors, i.e., $\underset{\sim}{F}' = (F_1, F_2, e_1, e_2, e_3, e_4)$. The structure is:

$$\Lambda = \begin{bmatrix} b_1^* & 0 & b_{e_1}^* & 0 & 0 & 0 \\ b_2^* & 0 & 0 & b_{e_2}^* & 0 & 0 \\ 0 & b_3^* & 0 & 0 & b_{e_3}^* & 0 \\ 0 & b_4^* & 0 & 0 & 0 & b_{e_4}^* \end{bmatrix} ,$$

and

$$\Phi = \begin{bmatrix} 1 & b_5^* & 0 & 0 & 0 & 0 \\ b_5^* & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \rho_{e_1 e_3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \rho_{e_1 e_3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

In contrast to previous formulations the error variances are standardized so that the correlations between e_1 and e_3 and F_1 and F_2 are estimated directly and in Λ the path coefficients of the observed variables on their errors $(b_{e_i}^*)$ are estimated. This model has 10 distinct elements in Σ and 10 parameters to be estimated $(b_1^*, b_2^*, b_3^*, b_4^*, b_{e_1}^*, b_{e_2}^*, b_{e_3}^*, b_{e_4}^*, b_5^*, \rho_{e_1 e_3})$, i.e., the model is just identified. The expected variance-covariance matrix $\Sigma = \Lambda\Phi\Lambda'$, i.e., the matrix Θ is taken to be zero.

The Figure 2.d. model poses two problems: the parameters b_3^*, b_4^*, and b_5^* are not identified and the expected correlation between X_4 and X_1 or X_2 is specified as zero even though the observed correlation may differ

from zero presumably because of sampling fluctuations. The analysis in Section II showed that X_4 does not contribute to the identification of parameters, i.e., only the product $b_3^* b_5^*$ is identified with or without X_4. Without X_4 the model is that of Figure 1.b. and no purpose is served by retaining F_2. Assuming all variables are standardized $X_1 = b_1^* F_1 + \theta_1$ may be substituted for $F_1 = b_1^* X_1 + e_1$ as noted earlier. With F_2 eliminated and knowing that only the correlation of X_3 with F_1 is identified the model may be written as:

$$X_1 = b_1^* F_1 + \theta_1 \quad , \tag{10a}$$

$$X_2 = b_2^* F_1 + e_2 \quad , \text{ and} \tag{10b}$$

$$X_3 = b_3^* b_5^* F_1 + b_3^* b_4^* X_4 + e_3' \quad \text{where} \quad e_3' = b_3^* \theta + e_3 \quad . \tag{10c}$$

For convenience define $b_{35}^* = b_3^* b_5^*$ and $b_{34}^* = b_3^* b_4^*$. For computational simplicity define a new factor x_4 which is identical to the observed X_4, i.e., $X_4 = x_4$. The factors are then $\underset{\sim}{F}' = (F_1, x_4)$,

$$\Lambda = \begin{bmatrix} b_1^* & 0 \\ b_2^* & 0 \\ b_{35}^* & b_{34}^* \\ 0 & 1 \end{bmatrix} \quad ,$$

$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & V_{x_4} \end{bmatrix} \quad ,$$

and

$$\Theta^2 = \begin{bmatrix} V_{\theta_1} & 0 & 0 & 0 \\ 0 & V_{e_2} & 0 & 0 \\ 0 & 0 & V_{e'_3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

In the fourth row of Θ^2 the diagonal cell is zero to indicate the identity $X_4 = x_4$ without residuals. If the expected matrix Σ is computed, i.e., $\Sigma = \Lambda\Phi\Lambda' + \Theta^2$ we find:

$$\Sigma = \begin{bmatrix} V_{X_1} & b^*_1 b^*_2 & b^*_1 b^*_{35} & 0 \\ b^*_1 b^*_2 & V_{X_2} & b^*_2 b^*_{35} & 0 \\ b^*_1 b^*_{35} & b^*_2 b^*_{35} & V_{X_3} & b^*_{34} \\ 0 & 0 & b^*_{34} & V_{X_4} \end{bmatrix} .$$

This shows that the expected correlations of X_1 and X_2 with X_4 are zero. This follows from the specification in Φ that x_4 is uncorrelated with F_1.

In the analysis of the model in Figure 2.e., the correlations among X_1, X_2, and F_1 were identified first and then b^*_1 and b^*_2 identified from these correlations. The simplest estimation procedure is to estimate the correlations among X_1, X_2, and F_1 and then compute b^*_1 and b^*_2 from the estimated correlations. This problem can be handled by defining two factors $x_1 = X_1$ and $x_2 = X_2$. The structural equations are:

$$X_1 = x_1 \quad ,$$

$$X_2 = x_2 \quad ,$$

$$X_3 = b_3^* F_1 + e_3 \quad , \text{ and}$$

$$X_4 = b_4^* F_1 + e_4 \quad .$$

The factors are $\underset{\sim}{F}' = (x_1 , x_2 , F_1)$,

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b_3^* \\ 0 & 0 & b_4^* \end{bmatrix} ,$$

$$\Phi = \begin{bmatrix} V_{x_1} & \rho_{x_1 x_2} & \rho_{x_1 F_1} \\ \rho_{x_1 x_2} & V_{x_2} & \rho_{x_2 F_1} \\ \rho_{x_1 F_1} & \rho_{x_2 F_1} & 1 \end{bmatrix} ,$$

and

$$\Theta^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & V_{e_3} & 0 \\ 0 & 0 & 0 & V_{e_4} \end{bmatrix} .$$

There are 10 distinct elements in Σ and nine parameters to be estimated ($b_3^*, b_4^*, V_{x_1}, V_{x_2}, \rho_{x_1 x_2}, \rho_{X_1 F_1}, \rho_{X_2 F_1}, V_{e_3}$, and V_{e_4}), so that the model has one

overidentifying restriction. Note that the estimated elements of Φ should be used to estimate b_1^* and b_2^* ($r_{X_1X_2}$ may not equal $\rho_{x_1x_2}$).

In relation to the model in Figure 2.f. Costner (1969) discussed the problem of ascertaining whether b_8^* was zero and of distinguishing the $b_8^* = 0$ model from one in which errors (e.g., e_3 and e_4) were correlated. To see how this is accomplished in Jöreskog's approach, first consider the model when $b_8^* = 0$ and treating residuals as factors:

$$\underset{\sim}{X}' = (X_1, X_2, X_3, X_4, X_5, X_6) \ ,$$

$$\underset{\sim}{F}' = (F_1, F_2, e_1, e_2, e_3, e_4, e_5, e_6) \ ,$$

$$\Lambda = \begin{bmatrix} b_1^* & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ b_2^* & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ b_3^* & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & b_4^* & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & b_5^* & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & b_6^* & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

$$\Phi = \begin{bmatrix} 1 & b_7^* & 0 & 0 & 0 & 0 & 0 & 0 \\ b_7^* & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{e_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{e_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{e_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{e_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{e_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{e_6} \end{bmatrix} ,$$

and

$$\Sigma = \Lambda\Phi\Lambda' \qquad (\text{i.e., } \Theta^2 = 0).$$

Note that we have chosen not to introduce the residual θ into the analysis because we wish to standardize both F_1 and F_2, in which case $\rho_{F_1F_2} = b_7^*$. This model is a variation of that in Figure 2.b. and all parameters are identified. There are 21 distinct elements in Σ and 14 parameters to be estimated so that there are seven overidentifying restrictions. To test $b_8^* \neq 0$, we specify $X_4 = b_8^*F_1 + b_4^*F_2 + e_4'$, i.e., in Λ the fourth row, first column element is left "free" instead of fixed = zero. This model has one more parameter to be estimated and therefore six overidentifying restrictions. Thus the original model is more restrictive and will therefore typically have a larger χ^2. In large samples, the difference in χ^2 between these two models, with degrees of freedom equal to the difference in number of restrictions, can be used to test the hypothesis that $b_8^* \neq 0$. Similarly the model with e_3 and e_4 correlated ("free") in Φ instead of independent (fixed = 0), would have six degrees of freedom and the difference in χ^2 with one degree of freedom would be a test of the hypothesis that e_3 and e_4 are uncorrelated. A comparison of the χ^2 for $b_8^* \neq 0$ to that for $\rho_{e_3e_4} \neq 0$ gives an indication of which is the better fitting model. Costner (1969, Figure 10) also raises the question of whether e_1 and e_2 are correlated. This hypothesis is tested by allowing the covariance between e_1 and e_2 in Φ to be "free," the change in χ^2 with one degree of freedom providing the appropriate statistical test. Hypotheses involving "constrained" parameters may be tested similarly, e.g., $b_1^* = b_6^*$ (Heise, 1969) or $V_{e_1} = V_{e_6}$ (Wiley & Wiley, 1970).

It can be observed that use of Jöreskog's program requires the investigator to know the identification status of each parameter, but does not require the complex algebraic manipulations provided by Costner (1969) and Blalock (1969). It is important to recognize the essentials of each model in order to fit it into Jöreskog's general model. Jöreskog's model assumes that the observed variables are "random" rather than "fixed" but it is doubtful that most applied sociologists need to be concerned about this issue which is minor in comparison to the usual questionable validity of measures and models.

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